Logarithmic Time Prediction

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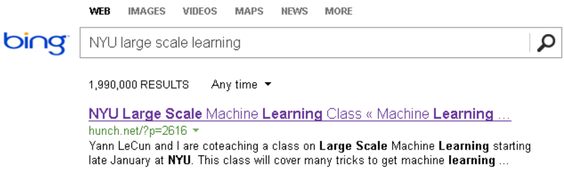
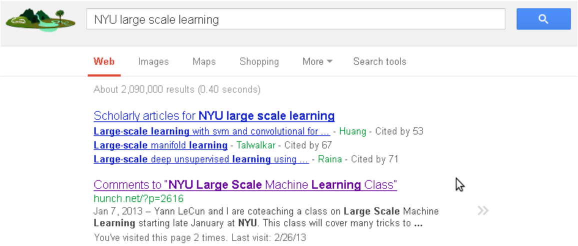
A Scenario

You have 10^{10} webpages and want to return the best result in 100ms.





How do you do it?

Who is that?

The Multiclass Prediction Problem

Repeatedly

1. See x
2. Predict $\hat{y} \in \{1, \ldots, K\}$
3. See y

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Goal: Find $h(x)$ minimizing error rate:

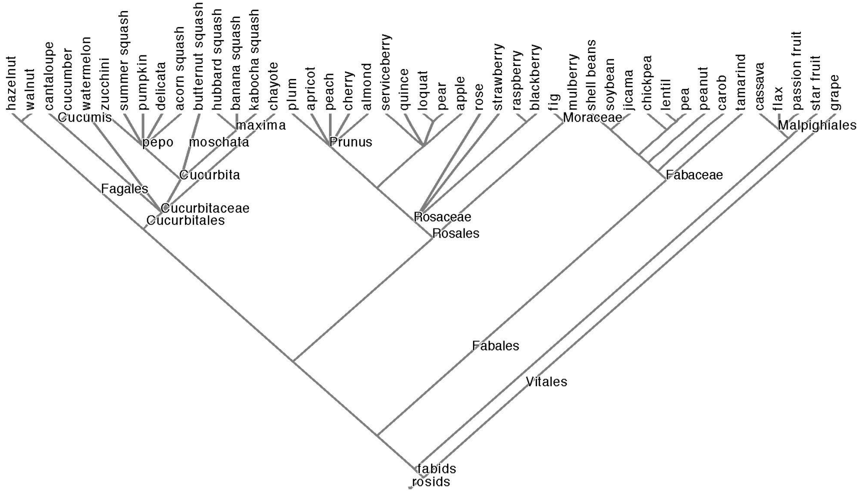
$$\Pr_{(x,y)\sim D}(h(x) \neq y)$$

with $h(x)$ fast.

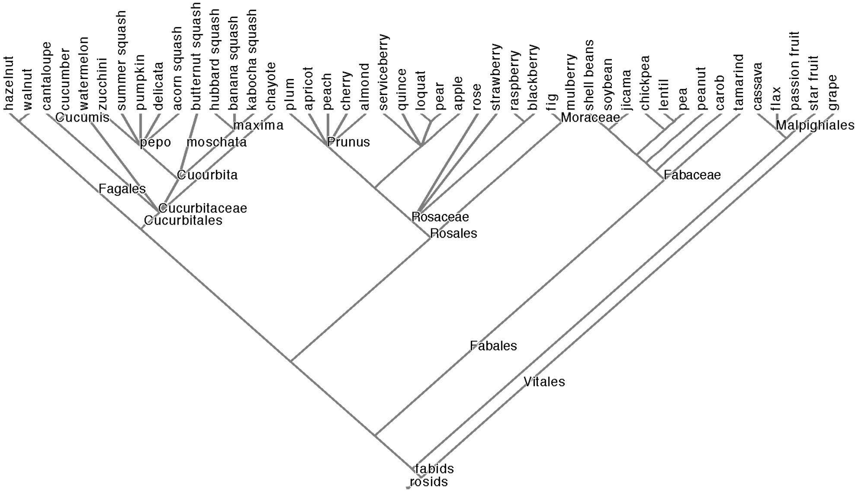
Trick #1

K is small

Trick #2: A hierarchy exists

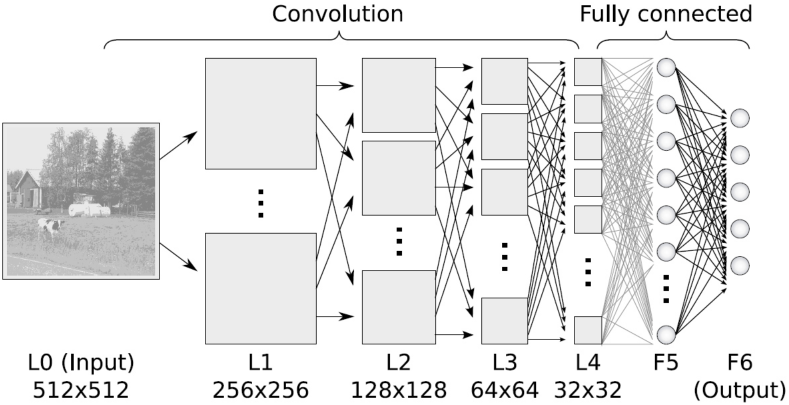


Trick #2: A hierarchy exists

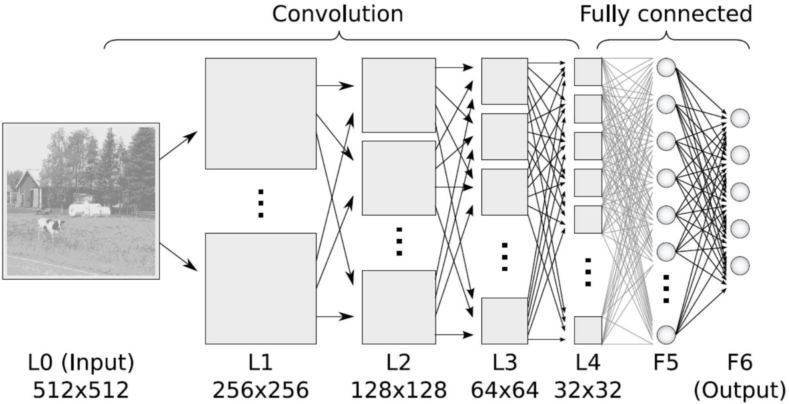


So use Trick #1 repeatedly.

Trick #3: Shared representation

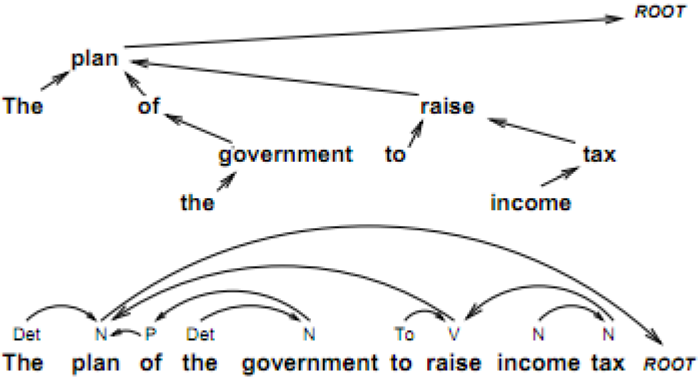


Trick #3: Shared representation

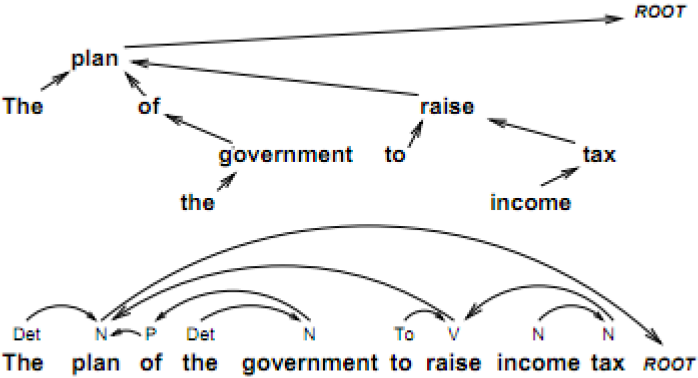


Very helpful... but computation in the last layer can still blow up.

Trick #4: "Structured Prediction"



Trick #4: "Structured Prediction"



But what if the structure is unclear?

Trick #5: GPU

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10 Teraflops is great... yet still burns energy.

Outline

1. Tricks
2. Static Structure
3. Dynamic Structure

How fast can we hope to go?

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Theorem: There exists multiclass classification problems where achieving 0 error rate requires $\Omega(\log K)$ time to train or test per example.

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Proof: Pick $y \sim U(1, ..., K)$

Any prediction algorithm outputting less than $\log_2 K$ bits loses with constant probability.

Any training algorithm reading an example requires $\Omega(\log_2 K)$ time.

Can we predict in time $O(\log_2 K)$?

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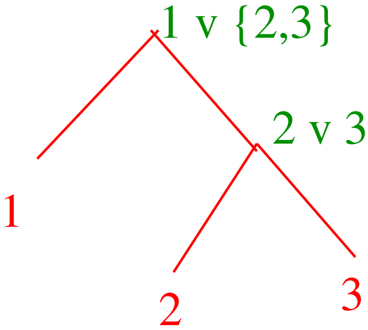
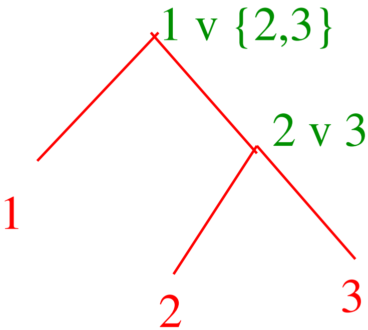
P(y=1) = .4
P(y=2) = .3
P(y=3) = .3



$P(\{2,3\}) > P(1) \Rightarrow$ lose for divide and conquer

Filter Trees [BLR09]

P(y=1) = .4
P(y=2) = .3
P(y=3) = .3



1. Learn 2v3 first
2. Throw away all error examples
3. Learn 1 v Survivors

Theorem: For all multiclass problems, for all binary classifiers, Multiclass Regret $\leq$ Average Binary Regret * $\log(K)$

What about with costs?

Cost-sensitive multi-class classification

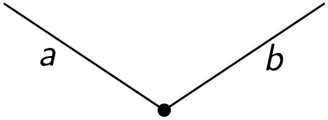
Distribution D over $X \times [0,1]^k$, where a vector in $[0,1]^k$ specifies the cost of each of the k choices.

Find a classifier $h : X \to \{1, \ldots, k\}$ minimizing the expected cost

$$\text{cost}(h, D) = \mathbf{E}_{(x,c)\sim D}[c_{h(x)}].$$

Generalization to the Cost-sensitive Case

To train a non-leaf node on example $(x, c_1, \dots, c_k)$:



$$\text{Let } y = \begin{cases} \text{left} & \text{if } c_a \le c_b \\ \text{right} & \text{otherwise} \end{cases}$$

Train on (x, y) with importance weight $|c_a - c_b|$.

Distribution induced at the node

Draw a cost-sensitive example from D, create an importance weighted sample as above.

Analysis

01 regret:
$\text{reg}_{01}(h, D) = \Pr_D(h(x) \neq y) - \min_{h'} \Pr_D(h'(x) \neq y)$
CS regret:
$\text{reg}_{CS}(h, D) = E_{(x,y)\sim D}[c_{h(x)}] - \min_{h'} E_{(x,y)\sim D}[c_{h'(x)}]$

Theorem

For all CSMC problems and node classifiers,

$$\text{reg}_{CS}(h_{FT}, D) \leq A(\text{reg}_{01}(h, D_{FT}))E_{D_{FT}} \sum_{\text{nodes } n} i_n$$

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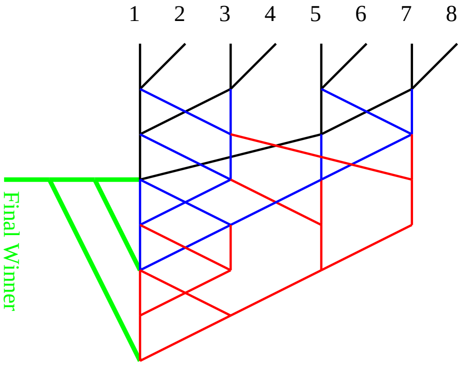
What's multiclass special case? Can you do better?

Error Correcting Tournament

Once an example loses, it moves to the next tournament. Once an example has lost e times, it is eliminated.

The e winners from the first phase compete in the final single elimination tournament. To win in round i, each player must defeat its opponent 2^{i-1} times.



$e = 3$

Summary of Multiclass results

	Filter Tree	Error Correcting Tour.
MC Comp.	$\log k$	$O(\log k)$
MC Regret	$\log k$	5.5
CS Train	k	$O(k \log k)$
CS Test	$\log k$	$O(\log k)$
CS Regret	$\min\{\frac{k}{2}, \sum_n i_n\}$	??

Contextual Bandits in Logarithmic time

Contextual Bandit Classification

Distribution D over $X \times [0,1]^k$, where a vector in $[0,1]^k$ specifies the cost of each of the k choices.

Find a classifier $h : X \to \{1, \ldots, k\}$ minimizing the expected cost

$$\text{cost}(h, D) = \mathbf{E}_{(x,c)\sim D}[c_{h(x)}].$$

given only observations $(x, a, c_a, p_a)^*$.

The Offset Tree for $k = 2$, $p = 0.5$

Suppose $k = 2$ for the moment and let $a \in \{-1, 1\}$. Create binary importance weighted samples according to:

$$\left(x,\ \text{sign}\left(a\left(\frac{1}{2} - c_a\right)\right),\ \left|\frac{1}{2} - c_a\right|\right)$$

x = context
$\text{sign}\left(a\left(\frac{1}{2} - c_a\right)\right)$ = label
$\left|\frac{1}{2} - c_a\right|$ = importance weight

Denoising Binary Importance Weighting

Theorem

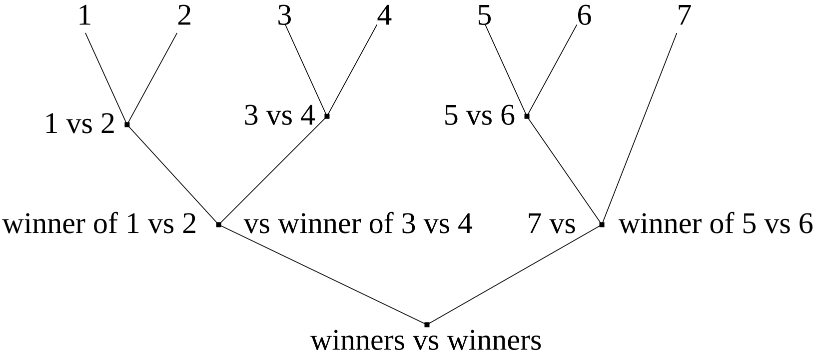
For all Contextual Bandit distributions D with $k = 2$, for all binary classifiers b:

$$\text{policy regret} \leq \text{reg}_{0/1}(b, D_{OT}).$$

Offset reduces noise in induced problem.

$\frac{1}{2}$ = minimax value of the median reward. Plugging in the actual median is always better.

Denoising for $k > 2$ arms



Use the same construction at each node. Internal nodes only get an example if all leaf-wards nodes agree with the label (Filtering trick).

Denoising with k arms

D_{OT} = Induced Binary classification problem
b = the classifier which predicts based on both x and the choice of binary problem according to D_{OT}.

Theorem

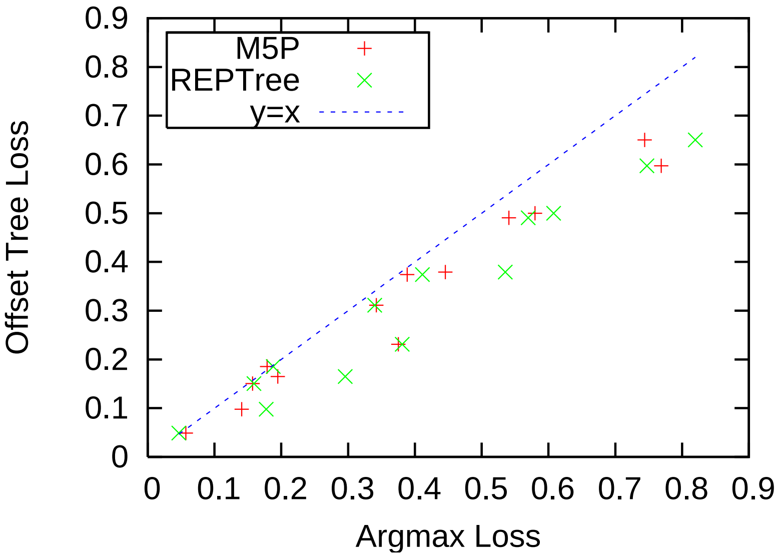
For all k-choice D, binary classifiers b:

$$\text{policy regret} \leq (k-1)\,\text{reg}_{0/1}(b, D_{OT}).$$

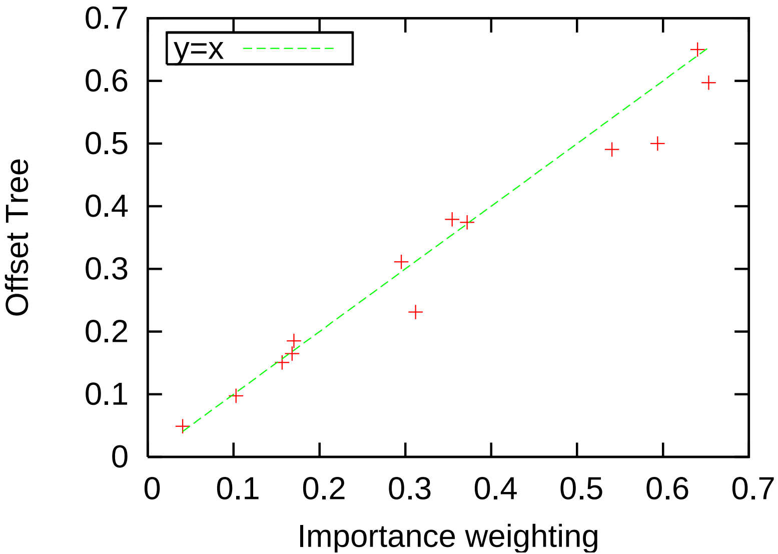
A Comparison of Approaches

Algorithm	Policy Regret Bound
Argmax	$\sqrt{2k \, \mathrm{reg}_{0/1}(s, D_{AR})}$
Importance Weighted	$4k \, \mathrm{reg}_{0/1}(b, D_{IW})$
Offset Tree	$(k-1) \, \mathrm{reg}_{0/1}(b, D_{OT})$

How do you expect things to work, experimentally?



M5P
REPTree
y=x
Offset Tree Loss
Argmax Loss
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9



y=x
Offset Tree
Importance weighting
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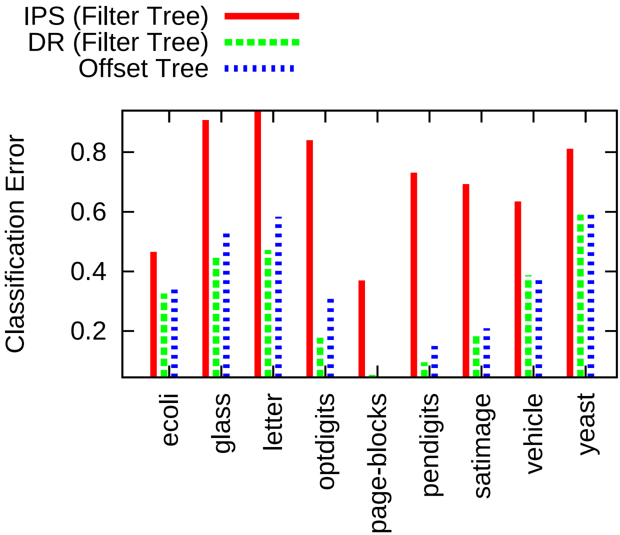
Compare with Double Robust



DR is exponentially slower, but often a bit better.

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How do you learn structure?

Not all partitions are equally difficult.
Compare $\{1,7\}v\{3,8\}$ to $\{1,8\}v\{3,7\}$
What is better?

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What is better?

[BWG10]: Better to confuse near leaves than near root.
Intuition: the root predictor tends to be overconstrained while the leafwards predictors are less constrained.

The Partitioning Problem [CL14]

Given a set of n examples each with one of K labels, find a partitioner h that maximizes:

$$E_{x,y} |\Pr(h(x) = 1, y) - \Pr(h(x) = 1) \Pr(y)|$$

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$$E_x \sum_y \Pr(y) \left| \Pr(h(x) = 1 | x \in X_y) - \Pr(h(x) = 1) \right|$$

where X_y is the set of x associated with y.

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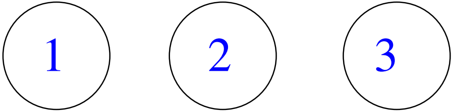
Nonconvex for any symmetric hypothesis class (ouch)

Bottom Up doesn't work



Suppose you use linear representations.

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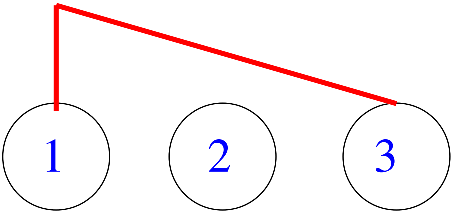

Suppose you use linear representations.
Suppose you first build a 1v3 predictor.

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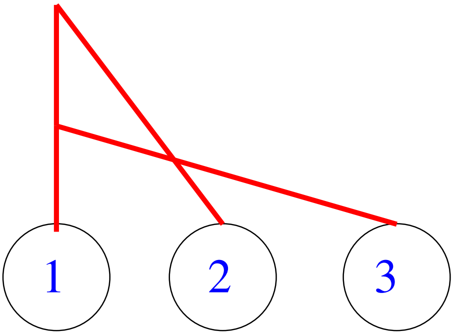

Suppose you use linear representations.
Suppose you first build a 1v3 predictor.
Suppose you then build a 2v{1v3} predictor.
You lose.

How do you train to partition?

Input: Example (x, y); Node n

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$$\hat{H}_{|\mathrm{left}} \doteq \frac{n.\mathrm{left.total}}{n.\mathrm{total}} \hat{H}'_{\mathrm{left}} + \frac{n.\mathrm{right.total}}{n.\mathrm{total}} \hat{H}_{\mathrm{right}}$$

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$\widehat{\Delta H}_{\mathrm{post}} \leftarrow \hat{H}_{|\mathrm{left}} - \hat{H}_{|\mathrm{right}}$

$\mathrm{Learn}_n(x, |\widehat{\Delta H}_{\mathrm{post}}|, \mathrm{sign}(\widehat{\Delta H}_{\mathrm{post}}))$

Important Optimizations

1. Do not descend to a pure leaf. Halt early and train one-against-some classifiers for each label.
2. Use large deviation bound on recall to control halting.
3. Add node id features as you descend the tree.

A boosting theorem

γ-Weak Learning Assumption

For all distributions $D(x, y)$ a learning algorithm using examples $(x, y)^*$ IID from D finds a binary classifier $c : X \rightarrow \{l, r\}$ satisfying

$$p_l H_l + p_r H_r \leq H_n - \gamma$$

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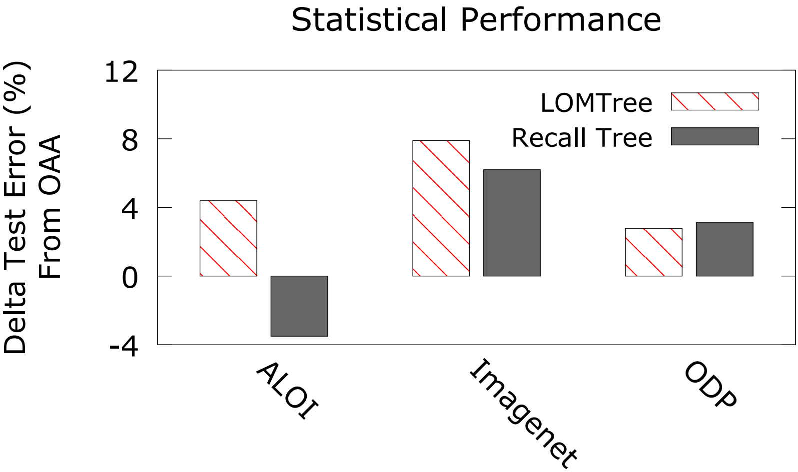
Theorem

If γ-Weak Learning holds, then after t splits the multiclass error rate ϵ of the tree is bounded by:

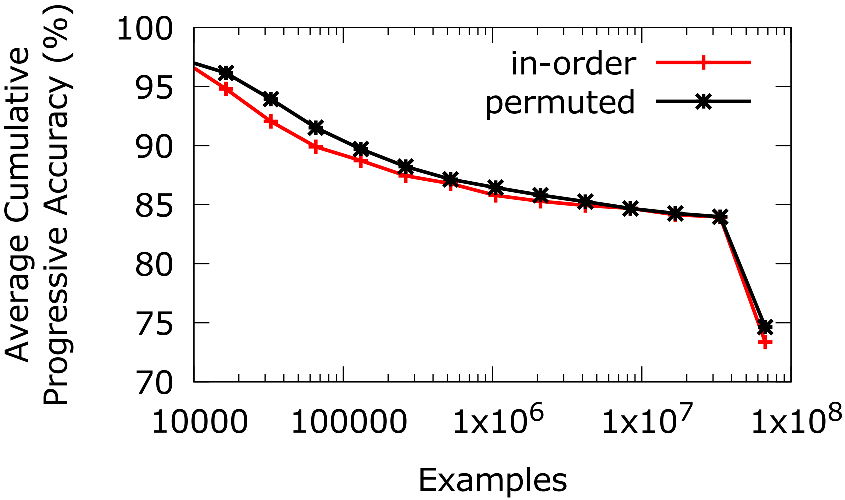
$$\epsilon \leq H_1 - \gamma \ln(t + 1)$$

where H_1 is the class label distribution entropy.

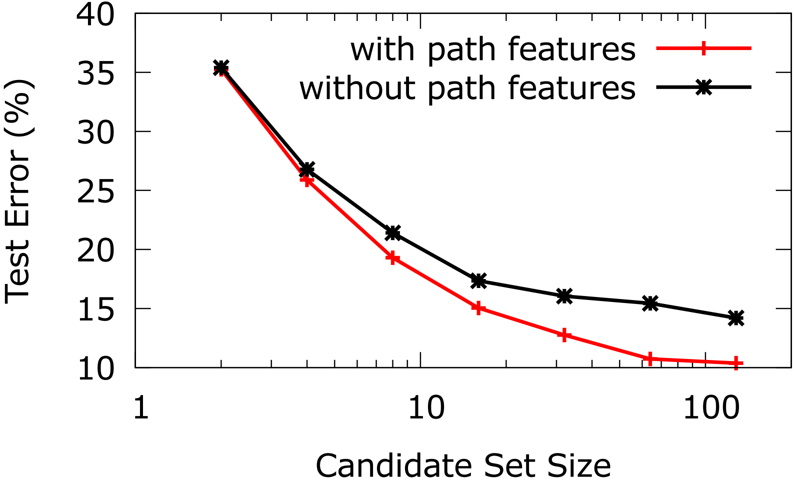
Accuracy vs. LOMtree, OAA



Online performance (LTCB)



With/without path features (ALOI)



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What is the right way to achieve consistency and dynamic partition?

How can you balance representation complexity and sample complexity?

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